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On steady recirculating flows

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Integral constraints are derived for steady recirculating flows of nearly incompressible fluids, arising from the action of a small amount of viscosity and heat conduction. These constraints are then combined with the inviscid non-diffusive incompressible flow equations to show that two-dimensional flows containing closed nested streamlines, or three-dimensional flows with closed nested stream surfaces, are isothermal. In the former case it is shown that the vorticity is constant, and in the latter case there is an analogous result when the flow is axially symmetric and confined to axial planes. For a circular cell free convection problem, the interior temperature and vorticity are determined from the boundary conditions by an approximate integration of the boundary layer equations.

1. Introduction

The equations governing the steady laminar motion of an inviscid non-diffusive incompressible fluid are

$$\operatorname{div}\mathbf{u} = 0, \tag{1.1}$$

$$\rho\mathbf{u}\times\mathbf{w} - (\tfrac{1}{2}q^2+\Omega)\rho\beta\nabla T - \nabla H = 0, \tag{1.2}$$

$$\mathbf{u}\,.\nabla T = 0, \tag{1.3}$$

where Ω is the gravitational potential and all the other symbols have their usual meanings (and are defined in the remarks following equations (2.3) and (2.9)). Except in the neighbourhood of certain singular stream surfaces, it will be assumed that these equations are valid as the Reynolds number approaches infinity (the Prandtl number is $O(1)$); in addition, the usual conditions that the fluid be incompressible are assumed to hold, viz. the Mach number is small, the length scale of the velocity field is small compared to the 'scale-height' of the atmosphere, and the variations of density due to viscous dissipation or heat conduction are small.

It is well known that equations (1.1), (1.2) and (1.3) are not sufficient to allow the velocity distribution to be determined from the specification of the normal velocity at prescribed boundaries. Indeed both T and H, which are constant along streamlines, may vary arbitrarily from one streamline to another. This indeterminacy can be removed when all the streamlines come from a region where both $\mathbf{u}$ and T are known (e.g. from a uniform state far upstream). However, for recirculating flows this method is not available. For flows with closed streamlines, we follow an approach due to Batchelor (1956), who considered fluids of *a priori* constant density, and consider the action of a small amount of viscosity and heat conduction; two integral constraints are obtained, which hold uniformly as the Reynolds number increases, and in the cases of two-dimensional flow, and axially symmetric flow without a swirl velocity, these constraints render the distribution of T and H determinate. In the two-dimensional case it is found that both the temperature and the vorticity are constant in a flow region containing nested closed streamlines, with analogous results in the axially symmetric case. For flows where the streamlines are not necessarily closed, but there exist closed stream surfaces, we adapt an idea due to Wood (1965), who considered fluids of *a priori* constant density, and again obtain two integral constraints. For a flow region containing nested, closed stream surfaces these constraints show that the temperature is constant, but are, in general, insufficient to determine completely the vorticity distribution.

In § 2 the integral constraints for closed streamline flows are derived, and in § 3 these are applied to the two-dimensional case. To determine the values of the constant temperature and vorticity in the inviscid, or core region, an appeal must be made to the boundary layer equations which hold in the vicinity of the singular streamline bounding the flow region. An approximate method of dealing with these equations which enables the core temperature and vorticity to be determined is outlined, and applied to a circular cell free convection problem, posed by Carrier (1953). In § 4 the integral constraints for flows containing closed stream surfaces are derived, and are applied to show that the core region has constant temperature, and that the stream surfaces are, in general, toroidal.

2. Derivation of integral constraints

We shall derive certain integral constraints for the steady recirculating flow of a fluid which is nearly inviscid, non-diffusive and incompressible. The equations of motion are

$$\operatorname{div}(\rho\mathbf{u}) = 0, \tag{2.1}$$

$$\rho\mathbf{u}\times\mathbf{w}+(\tfrac{1}{2}q^2+\Omega)\,\nabla\rho-\nabla H = -\mu(\tfrac{4}{3}\operatorname{grad}\operatorname{div}\mathbf{u}-\operatorname{curl}\mathbf{w}), \tag{2.2}$$

$$\rho T\mathbf{u}\,.\,\nabla S = k\nabla^2 T+\Phi, \tag{2.3}$$

where ρ, $\mathbf{u}$, T, S are the density, velocity, temperature and entropy respectively of a fluid particle; $\mathbf{w} = \operatorname{curl}\mathbf{u}$, $q = |\mathbf{u}|$ and $H = \frac{1}{2}\rho q^2+p+\rho\Omega$ is the local 'total head' in the fluid, where p is the pressure and Ω is the gravitational potential; μ is the coefficient of shear viscosity and k the thermal conductivity, and both are assumed constant; finally Φ is the viscous dissipation function. The flow is characterized by a length scale L, a velocity scale U, a density scale ρ_0, a temperature difference scale ΔT_0, and a local speed of sound scale c_0; from these we form the dimensionless parameters

$$M^2 = U^2/c_0^2,\quad F = U^2/Lg,\quad R = \rho_0 UL/\mu,\quad E = U^2/c_p\Delta T_0,\quad \sigma = \mu c_p/k, \tag{2.4}$$

where g is the acceleration due to gravity, and c_p is the specific heat at constant pressure and is assumed constant. We shall assume that the Prandtl number σ

is $O(1)$, which is the case for most common liquids and gases. The parameters M^2, M^2/F, $1/R$ and E are all assumed to be small, and are taken as zero in the inviscid non-diffusive incompressible limit.

In the energy equation (2.3) the viscous dissipation function Φ is $O(E)$ compared to the heat conduction term $k\nabla^2 T$, and will henceforth be neglected; indeed the Eckert number E is $O((T_0/\Delta T_0) M^2)$ where T_0 is a temperature scale and, for the length scales encountered in a laboratory, is usually much smaller than $1/R$ (for the length scales encountered in the atmosphere E could be $O(1/R)$). We suppose that $\mathscr{C}$ is a closed streamline; then from (2.3)

$$\oint_{\mathscr{C}} \mathbf{u} \,.\, \nabla S \frac{dl}{q} = k \oint_{\mathscr{C}} \frac{\nabla^2 T}{\rho T} \frac{dl}{q}, \tag{2.5}$$

and as S is single-valued, the integral on the left vanishes. Hence, for each closed streamline $\mathscr{C}$, we have the integral constraint

$$\oint_{\mathscr{C}} \frac{\nabla^2 T}{\rho T} \frac{dl}{q} = 0, \tag{2.6}$$

which holds for any non-zero value of k. If the streamline remains closed as the non-diffusive limit is approached, and remains in the region within which this limit is valid, then the constraint (2.6) holds also in this limit.

To obtain a condition on the vorticity distribution, analogous to (2.6), we will assume that the fluid is *a priori* incompressible (i.e. $\operatorname{div} \mathbf{u} = 0$). The fluid will be nearly incompressible when the parameters M^2, M^2/F and $(\Delta T_0/T_0)(1/R)$ are all small (e.g. Batchelor 1967, § 3.6); here we assume that these parameters are not only small, but small compared with $1/R$ (this is the case for most liquids, and for most gases when the length scale is that typically encountered in a laboratory). The equations of motion become:

$$\operatorname{div} \mathbf{u} = 0, \tag{2.7}$$

$$\rho \mathbf{u} \times \mathbf{w} - (\tfrac{1}{2} q^2 + \Omega) \rho \beta \nabla T - \nabla H = \mu \operatorname{curl} \mathbf{w}, \tag{2.8}$$

$$\rho c_p \mathbf{u} \,.\, \nabla T = k \nabla^2 T, \tag{2.9}$$

where we are again ignoring the viscous dissipation function Φ; here, β is the coefficient of thermal expansion (assumed constant) and we have used the thermodynamic equation of state

$$\nabla \rho + \rho \beta \nabla T = 0, \tag{2.10}$$

which is valid under the conditions of incompressibility given above. Equations (2.8) and (2.9) imply that

$$\mu (\tfrac{1}{2} q^2 + \Omega)(\beta/\sigma) \nabla^2 T + \mathbf{u} \,.\, \nabla H = -\mu \mathbf{u} \,.\, \operatorname{curl} \mathbf{w}, \tag{2.11}$$

and integrating around the closed streamline $\mathscr{C}$, we obtain

$$\frac{\beta}{\sigma} \oint_{\mathscr{C}} (\tfrac{1}{2} q^2 + \Omega) \nabla^2 T \frac{dl}{q} = - \oint_{\mathscr{C}} \mathbf{u} \,.\, \operatorname{curl} \mathbf{w} \frac{dl}{q}, \tag{2.12}$$

as H is single-valued. Now σ is $O(1)$, and further is a fluid property; thus, if the streamline remains closed as $1/R \to 0$ (i.e. as $k, \mu \to 0$), and remains in the region where this limit is valid, then the constraint holds also in this limit. In practice, the term containing q^2 is much smaller than either of the other two terms, and may be omitted. However, the term $\mathbf{u}.\mathrm{curl}\,\mathbf{w}$ is $O(\Delta T_0/T_0 . 1/F)$ compared with $\beta\Omega\nabla^2 T$, and so these two terms are usually comparable.

We have been unable to obtain a condition analogous to (2.12) when the fluid is *a priori* compressible. Although (2.2) may be written in the form

$$\rho\mathbf{u}\times\mathbf{w}+\rho T\nabla S-\rho\nabla(\tfrac{1}{2}q^2+\Omega+E+p/\rho) = -\mu(\tfrac{4}{3}\,\mathrm{grad\,div}\,\mathbf{u}-\mathrm{curl}\,\mathbf{w}), \quad (2.13)$$

where E is the specific internal energy of a fluid particle, so that, using (2.3) with Φ omitted,

$$\mu.c_p/\sigma\nabla^2 T-\rho\mathbf{u}.\nabla(\tfrac{1}{2}q^2+\Omega+E+p/\rho) = -\mu\mathbf{u}.(\tfrac{4}{3}\,\mathrm{grad\,div}\,\mathbf{u}-\mathrm{curl}\,\mathbf{w}), \quad (2.14)$$

we must now reject the term on the right-hand side of (2.14) as it is $O(E)$ compared with the first term on the left-hand side. Our procedure then yields a condition indistinguishable from (2.6) in the non-diffusive limit.

3. Two-dimensional flow with closed streamlines

In the inviscid non-diffusive incompressible limit the equations of motion become (1.1), (1.2), (1.3) and (2.10). When the flow is two-dimensional we can introduce a stream function ψ, and use (ψ, ξ) as orthogonal curvilinear co-ordinates where the curves ξ = constant are everywhere orthogonal to streamlines. The displacements corresponding to increments in ψ and ξ are $d\psi/q$ and $h\,d\xi$ where h is unknown. Then equation (1.3) implies that T is constant on streamlines (and also ρ is constant on streamlines), and so $T = T(\psi)$.

We suppose that the flow region, for which the limit $k, \mu \to 0$ applies, contains a set of closed nested streamlines whose inner boundary is merely a point. Then the constraint (2.6) implies that

$$\frac{d}{d\psi}\left(\frac{dT}{d\psi}\oint_{\mathscr{C}} qh\,d\xi\right) = 0. \quad (3.1)$$

Assuming that q is finite everywhere, we integrate to find that

$$\frac{dT}{d\psi} = 0 \quad \text{and} \quad T = T_0\,(\text{constant}) \quad (3.2)$$

everywhere in the flow region. From (2.10) it follows that the density is also constant everywhere. Then equation (1.2) implies that H is constant on streamlines and so $H = H(\psi)$, whence $\rho w = dH/d\psi$ where w is the component of vorticity normal to the flow region (and is the only non-zero component). Assuming that the constraint (2.12) obtains, we see from (3.2) that the left-hand side vanishes, and it follows that (as in Batchelor 1956)

$$\frac{dw}{d\psi}\oint qh\,d\xi = 0. \quad (3.3)$$

Hence the vorticity is constant everywhere.

The condition (3.2) has been previously obtained by Burggraf (1966), under the assumption that the fluid is *a priori* of constant density; he used an integral constraint obtained from (2.9), and rejected the contribution from the dissipation function by assuming that the flow was a perturbation of a solid-body rotation. Weinbaum (1964) also obtained the condition (3.2); he used the integral constraint of zero heat flux across each closed streamline.

The value of the constant T_0 (and of the constant vorticity w_0) is undetermined as yet, and clearly cannot be determined from a consideration of the inviscid, non-diffusive region, or core region, alone. We shall assume that the core region is surrounded by a singular streamline, $\mathscr{C}_0$, on which the temperature and velocity are specified, and in the neighbourhood of which the limit $k, \mu \rightarrow 0$ applies. If $\mathscr{C}_0$ is the streamline $\psi = 0$, the core equations are

$$T = T_0, \quad \nabla^2\psi = w_0 \quad \text{where} \quad \psi = 0 \quad \text{on} \quad \mathscr{C}_0. \tag{3.4}$$

We let q_0 denote $\partial\psi/\partial n\,|_{\mathscr{C}_0}$, which is the velocity on $\mathscr{C}_0$ determined from the solution of (3.4) ($\mathbf{n}$ is the outward normal to $\mathscr{C}_0$). These equations are to be supplemented by boundary layer equations in the vicinity of $\mathscr{C}_0$. Since σ is $O(1)$, the velocity and thermal boundary layers are of comparable depths, viz. $O(L/\sqrt{R})$ where L is a typical dimension of $\mathscr{C}_0$. We shall write these equations in their intrinsic form (q.v. Howarth, 1969, §§ 3 and 27),

$$\frac{1}{2}\frac{\partial}{\partial\xi}(q^2 - q_0^2) - \beta(T - T_0)\frac{\partial\Omega_0}{\partial\xi} = \tfrac{1}{2}\nu h_0 q \frac{\partial^2}{\partial\psi^2}(q^2), \tag{3.5}$$

$$\frac{\partial T}{\partial\xi} = \frac{\nu}{\sigma} h_0 \frac{\partial}{\partial\psi}\left(q\frac{\partial T}{\partial\psi}\right), \tag{3.6}$$

where Ω_0, h_0 are the values of Ω, h respectively on $\mathscr{C}_0$, and ν is the kinematic viscosity u/ρ_0. Then if $\mathscr{C}$ is a closed streamline, lying within the boundary layer, we obtain from the integration of (3.5) and (3.6)

$$\frac{\partial}{\partial\psi}\oint h_0 q \frac{\partial T}{\partial\psi} d\xi = 0, \tag{3.7}$$

$$\frac{\partial}{\partial\psi}\frac{\beta}{\sigma}\oint \Omega_0 h_0 q \frac{\partial T}{\partial\psi} d\xi = \frac{1}{2}\oint h_0 q \frac{\partial^2}{\partial\psi^2}(q^2)\, d\xi. \tag{3.8}$$

It would seem that, before we can integrate (3.7) and (3.8), the boundary-layer equations (3.5) and (3.6) must be solved. Being unable to do this exactly, we linearize these equations by replacing the q on the right-hand side by cq_0, where c is a constant to be determined (applied to (3.5) with $T = T_0$ and $c = 1$, this is the Kármán and Millikan approximation). Making the same change in (3.7) and integrating, we obtain

$$\oint_{\mathscr{C}} h_0 q_0 T\, d\xi = \text{constant}, \tag{3.9}$$

where one constant of integration has been evaluated by taking $\mathscr{C}$ at the outer edge of the boundary layer. Evaluating (3.9) on $\mathscr{C}_0$, and on the outer edge of the boundary layer, we obtain, to the accuracy of the boundary layer approximation,

$$T_0\oint_{\mathscr{C}_0} h_0 q_0\, d\xi = \oint_{\mathscr{C}_0} h_0 q_0 T\, d\xi. \tag{3.10}$$

When the same approximation is used on (3.8), it may also be integrated to yield the condition

$$\frac{\beta}{\sigma}T_0\oint_{\mathscr{C}_0}\Omega_0 h_0 q_0\,d\xi-\frac{\beta}{\sigma}\oint_{\mathscr{C}_0}\Omega_0 h_0 q_0 T\,d\xi=\frac{1}{2}\oint_{\mathscr{C}_0}h_0 q_0^3\,d\xi-\frac{1}{2}\oint_{\mathscr{C}_0}h_0 q_0 q^2\,d\xi. \quad (3.11)$$

Equations (3.10) and (3.11) allow T_0 and w_0 to be determined from the prescribed values of T and q on $\mathscr{C}_0$.

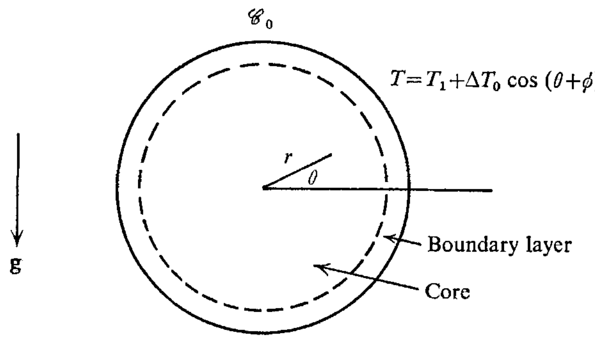

FIGURE 1. Geometry of the circular cell convection problem.

The procedure will be illustrated by considering free convection in a circular cell, a problem introduced by Carrier (1953) and discussed in detail by Weinbaum (1964). Using the polar co-ordinates (r, θ) as described in figure 1, the prescribed conditions on the boundary $r = a$, are $\mathbf{u} = 0$ and $T = T_1+\Delta T_0\cos(\theta+\phi)$ for $0 < \phi \leqslant \frac{1}{2}\pi$. Then the solution of (3.4) is the solid-body rotation

$$\psi = \tfrac{1}{4}w_0(r^2-a^2). \quad (3.12)$$

Hence
$$q_0 = \tfrac{1}{2}w_0 a. \quad (3.13)$$

Equation (3.10) shows that $T_0 = T_1$ (an obvious conclusion in this problem), and (3.11) then shows that

$$w_0 = 2\left(\frac{g\beta\Delta T_0 \sin\phi}{\sigma a}\right)^{\frac{1}{2}}. \quad (3.14)$$

Weinbaum treated this problem by using a 'modified' Oseen linearization on the boundary layer equations. For the case $\phi = \frac{1}{2}\pi$, he finds that w_0 is $1/\sqrt{2}$ times the value given by (3.14) in the 'classical' Oseen approximation, and a factor of at least two greater than this in the 'modified' Oseen approximation. Equation (3.14) implies that $w_0 \to 0$ as $\phi \to 0$; as this result is incompatible with the assumption of a non-trivial core motion on which our theory is based, the case $\phi = 0$ must be excluded. However, Weinbaum's analysis also predicts that $w_0 = 0$ when $\phi = 0$, and further evidence that the core is stagnant when $\phi = 0$ is provided by the experiments of Elder (1965) on the side-to-side heating of a vertical rectangular slot, although he used a Prandtl number considerably larger than that envisaged here. (Gill (1966) has given a theoretical analysis of this problem, and Elder (1966) has given a numerical analysis.) Clearly, when the core is stagnant, the core temperature need not be uniform; indeed Elder found a constant vertical temperature gradient. Fromm (1965), Burggraff (1966) and Robinson (1967) have each given numerical studies of convection in rectangular cells; for non-stagnant cores their calculations would appear to confirm that the core region is one of uniform temperature and vorticity.

For a steady, axially symmetric flow without azimuthal swirl, so that the flow takes place in axial planes, a very similar treatment to the preceding may be given. Indeed, if there is a flow region containing a set of closed nested streamlines, and for which the limit $k, \mu \to 0$ applies the condition (2.6) implies that the region is one of constant temperature. If the constraint (2.12) also obtains, then it follows (Batchelor 1956) that the vorticity is a linear function of distance from the axis of symmetry.

4. Flow with closed stream surfaces

A recirculating flow is unlikely to possess a set of nested, closed streamlines unless it possesses some special symmetry, such as that imposed by assuming that the flow is two-dimensional, or axially symmetric without a swirl velocity. We therefore characterize the flow as one containing a nested set of closed stream surfaces, and seek constraints analogous to (2.6) and (2.12) for such a flow. Each closed stream surface $\mathscr{S}$ is specified by the parameter α (e.g. the volume enclosed by $\mathscr{S}$). Then equation (2.3), with the dissipation function Φ omitted, is integrated over the volume V between two closed stream surfaces, $\mathscr{S}_1$ and $\mathscr{S}_2$ ($\mathscr{S}_2$ containing $\mathscr{S}_1$):

$$k\iiint_V \frac{\nabla^2 T}{T} dV = \iint_{\mathscr{S}_2} S\rho\mathbf{u}\,.\,\mathbf{n}\,dA - \iint_{\mathscr{S}_1} S\rho\mathbf{u}\,.\,\mathbf{n}\,dA, \tag{4.1}$$

where $\mathbf{n}$ is the outward normal, and we have used the equation $\operatorname{div}(\rho\mathbf{u}) = 0$. The right-hand side of (4.1) now vanishes as $\mathscr{S}_1$, $\mathscr{S}_2$ are stream surfaces, and differentiating the left-hand side with respect to α, we obtain the integral constraint

$$\iint_{\mathscr{S}} \frac{\nabla^2 T}{T} \frac{dA}{|\nabla\alpha|} = 0, \tag{4.2}$$

which holds for any non-zero value of k. If the stream surface remains closed as the non-diffusive limit is approached, and remains in the region where this limit is valid, then the constraint (4.2) holds in this limit also. In the non-diffusive, incompressible limit the constant temperature surfaces (which may also be identified as constant entropy surfaces in this limit) are also stream surfaces by (1.3). It will be assumed in the sequel that the surfaces $\mathscr{S}$ become constant temperature surfaces as $k \to 0$. An alternative method of characterizing the flow is to assume that the surfaces $\mathscr{S}$ are, *a priori*, constant entropy surfaces, and become stream surfaces only in the limit $k \to 0$. Then (4.1) still holds, and if there are no sources or sinks in the volume enclosed by any $\mathscr{S}$, then the right-hand side again vanishes, and (4.2) still holds.

In either case, $T = T(\alpha)$ in the non-diffusive, incompressible limit. We let η^1, η^2 be co-ordinates on $\mathscr{S}$, so that (4.2) becomes

$$\iint \sum_{r=1}^{3} \frac{\partial}{\partial \eta^r}\left\{\sqrt{g}\, g^{r3} \frac{dT}{d\alpha}\right\} d\eta^1 d\eta^2 = 0, \tag{4.3}$$

where $\eta^3 = \alpha$, g^{rs} are the contravariant components of the metric tensor of the co-ordinate system (η^1, η^2, η^3), and $g^{-1} = \det[g^{rs}]$. Since $\mathscr{S}$ is a closed surface η^1, η^2 are periodic co-ordinates, and there is no contribution to the integral from $r = 1$ or $r = 2$. Hence (4.3) becomes

$$\frac{d}{d\alpha}\left(\frac{dT}{d\alpha}\iint \sqrt{g g^{33}}\, d\eta^1\, d\eta^2\right) = 0, \tag{4.4}$$

and so we find that $$\frac{dT}{d\alpha} = 0 \quad \text{and} \quad T = T_0 \text{ (constant)} \tag{4.5}$$

everywhere in the flow region (which is now assumed to have no inner boundary). From (2.10) it follows that the density is also constant everywhere.

Next, we seek a constraint on the vorticity distribution analogous to (2.12). As in § 2, the fluid is again taken as *a priori* incompressible, and if V is the volume between two closed stream surfaces $\mathscr{S}_1$ and $\mathscr{S}_2$, we obtain from (2.11)

$$\mu \,.\, \frac{\beta}{\sigma}\iiint_V (\tfrac{1}{2}q^2 + \Omega)\,\nabla^2 T\, dV + \mu \iiint_V \mathbf{u}\,.\,\mathrm{curl}\,\mathbf{w}\, dV = -\iint_{\mathscr{S}_2} H\mathbf{u}\,.\,\mathbf{n}\, dA + \iint_{\mathscr{S}_1} H\mathbf{u}\,.\,\mathbf{n}\, dA. \tag{4.6}$$

As above, the right-hand side of (4.6) vanishes, and so

$$\frac{\beta}{\sigma}\iint (\tfrac{1}{2}q^2 + \Omega)\,\nabla^2 T \frac{dA}{|\nabla\alpha|} = -\iint \mathbf{u}\,.\,\mathrm{curl}\,\mathbf{w}\frac{dA}{|\nabla\alpha|}, \tag{4.7}$$

which holds for any value of μ, and under the conditions given above, holds also in the limit $\mu \to 0$. In this limit the constant H surfaces are also stream surfaces, and it will be assumed that the surfaces become constant H surfaces as $\mu \to 0$. Alternatively, following Wood (1965), we may assume that the surfaces $\mathscr{S}$ are, *a priori*, constant H surfaces, and become stream surfaces only in the limit $\mu \to 0$. Then (4.6) and (4.7) still hold, if there are no sources or sinks in the volume enclosed by any $\mathscr{S}$.

If, in the non-diffusive inviscid incompressible limit, it is assumed that the constant H surfaces are also constant temperature surfaces (this is the case, e.g. if the fluid was initially at rest q.v. Yih (1965), chapter 1, § 5), then it follows from (4.5) that the left-hand side of (4.7) is zero. Then, since

$$\mathbf{u} \times \mathbf{w} = \nabla\mathscr{H}, \tag{4.8}$$

where $\mathscr{H} = H/\rho_0 = \mathscr{H}(\alpha)$, we have

$$\iint_{\mathscr{S}} |\mathbf{w}|^2 \frac{dA}{|\nabla\alpha|} = \iint_{\mathscr{S}} \nabla^2\mathscr{H}\frac{dA}{|\nabla\alpha|}. \tag{4.9}$$

Hence $$\iiint_V |\mathbf{w}|^2\, dV = \frac{d\mathscr{H}}{d\alpha}\iint_{\mathscr{S}} |\nabla\alpha|\, dA, \tag{4.10}$$

where V is the volume enclosed by $\mathscr{S}$. Thus if the flow is genuinely rotational, so that the left-hand side of (3.10) is nowhere zero, it follows that $\nabla\mathscr{H}$ is nowhere zero, and the flow has no stagnation points. Then since the stream surfaces are smooth, and bounded, it follows from well-known theorems of topology (e.g.

Alexandroff & Hopf, 1935, ch. XIV, §4) that $\mathscr{S}$ is a toroid (i.e. topologically equivalent to a torus). This observation was made by Kruskal & Kulsrud (1958) in the context of the magnetostatic equations. Equation (4.10) provides one condition to determine the two vorticity fluxes (azimuthal and longitudinal) within the toroid. In general, therefore, a second condition is needed, analogous to that obtained by Wood (1965) for a fluid of *a priori* constant density.

The Euler equation (3.8) has recently received extensive discussion in the literature on the static equilibrium of conducting fluids (for a recent survey, q.v. Grad 1967). Indeed if $\mathbf{u}$ is identified as the magnetic field, $\mathbf{w}$ as the electric current density, and $\mathscr{H}$ as the fluid pressure, then there is an exact analogy with the equations describing static equilibria of plasmas. In seeking existence theorems for these equations, it has been found necessary to solve 'magnetic' differential equations of the type

$$\mathbf{u}\,.\,\nabla r = s \tag{4.11}$$

on a toroid, where $\mathbf{u}$, s are given and r is to be determined. Grad (1967), using a lemma of Moser's, has shown that a necessary and sufficient condition for the existence of a smooth, possibly multi-valued r is that $\oint s(dl/q)$ be a function of α only for each closed streamline lying on a given toroid $\mathscr{S}$ (see also Newcomb 1959 and Hamada 1962). Grad has suggested that this condition is implausible, and that, except for geometries of special symmetries, the Euler equation (3.8) has no smooth solutions on toroidal stream surfaces. If this conjecture is correct, it would imply that, in general, there are no steady, inviscid flows with toroidal stream surfaces.

It is a pleasure to record many interesting discussions with Dr W. W. Wood, who has stimulated my interest in this topic.

REFERENCES

Alexandroff, P. & Hopf, H. 1935 *Topologie*. Berlin: Springer.

Batchelor, G. K. 1956 *J. Fluid Mech.* **1**, 177.

Batchelor, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

Burggraf, O. R. 1966 *J. Fluid Mech.* **24**, 113.

Carrier, G. F. 1953 *Adv. appl. Mech.* **3**, 1.

Elder, J. W. 1965 *J. Fluid Mech.* **23**, 77.

Elder, J. W. 1966 *J. Fluid Mech.* **24**, 823.

Fromm, J. E. 1965 *Phys. Fluids*, **8**, 1757.

Gill, A. E. 1966 *J. Fluid Mech.* **26**, 515.

Grad, H. 1967 *Phys. Fluids*, **10**, 137.

Hamada, S. 1962 *Nuclear Fusion*, **2**, 23.

Howarth, L. 1959 In *Handbuch der Physik*, vol. VIII/1.

Kruskal, M. D. & Kulsrud, R. M. 1958 *Phys. Fluids*, **1**, 265.

Newcomb, W. A. 1959 *Phys. Fluids*, **2**, 362.

Robinson, J. L. 1967 *J. Fluid Mech.* **30**, 577.

Weinbaum, S. 1964 *J. Fluid Mech.* **18**, 409.

Wood, W. W. 1965 *J. Fluid Mech.* **22**, 337.

Yih, C. S. 1965 *Dynamics of Non-homogeneous Fluids*. New York: Macmillan.